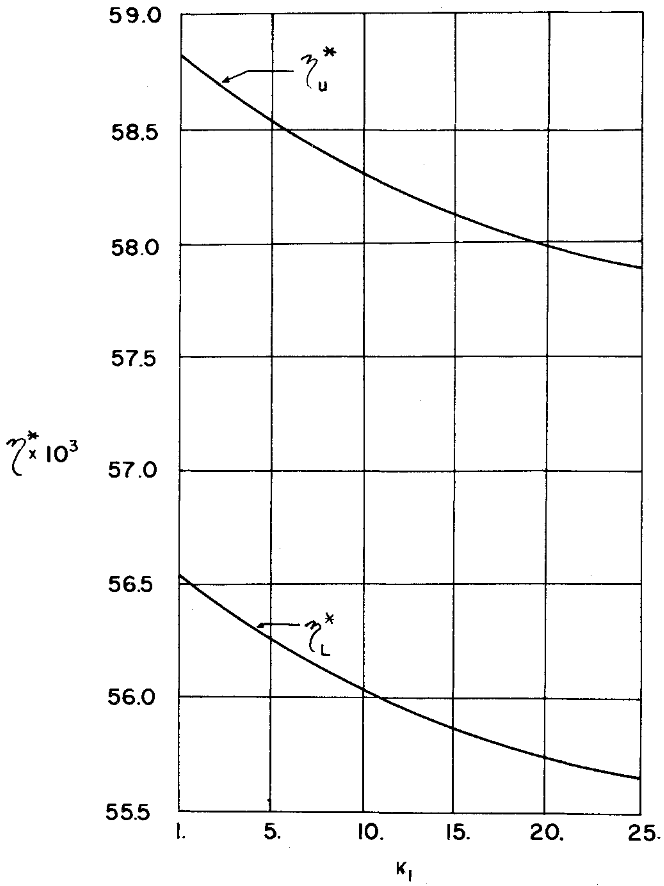

Fig. 2 Coefficients for uniaxial buckling.

For $k=1$, i.e., for the case of isotropic core, the critical loads N^*_u and N^*_L, given by Eqs. (16) and (17) reduce to those obtained by Fulton.[1]

In order to show the effect of orthotropic core upon the critical loads, the following numerical values for shell property and dimensions are used: $E_I=E_2=E$, $t_I=t_2=t=0.05$ in., $\mu=0.3$, $E/G_{xz}=2\times10^3$, $a=100$ in., $R=200$ in., $C=0.5$ in. The critical loads are expressed in the following form as

$$N^*_u=\eta^*_u(\pi^2D/a^2) \tag{18}$$

$$N^*_L=\eta^*_L(\pi^2D/a^2) \tag{19}$$

where η^*_u and η^*_L are the load coefficients. The values assumed for k_l ranged from 1-25 at intervals of 5. The curves are shown in Fig. 2. It can be seen from Fig. 2 that as k_l increases, the values for η^*_u and η^*_L decrease. The η^*-values show a maximum average of 0.42% decrease in the first k_l interval and the average rate of 0.29% decrease in the last k_l interval.

IV. Discussion and Conclusions

The nonlinear differential equations for shallow sandwich shells with orthotropic cores are derived herein. An example illustrating the use of these equations is presented. The approximate solution used in solving the example problem satisfies the boundary conditions for w, ϕ, M_x, and M_y given in Eqs. (8) and (9); however, the boundary conditions on F given in Eqs. (8) and (9) are satisfied only on the average. It should be noted that Fig. 2 represents load coefficients for a particular shell property and dimensions. The assumed deflection shape given by Eq. (10) for a square shallow shell represents the buckling mode shape of a half sine wave in both x and y directions. For a more general case of shell dimensions and property, the deflection function capable of representing any number of half-waves in both x and y directions should be used instead of Eq. (10), and the critical loads should be obtained from Eqs. (15) and (17).

References

[1] Fulton, R. E., "Nonlinear Equations for Shallow Unsymmetrical Sandwich Shell of Double Curvature," *Proceedings of the 7th Midwestern Mechanics Conference, Developments in Mechanics*, Vol. 1, 1961, pp. 365-380.

[2] Ronan, J. G., "Large Deflection of Shallow Sandwich Shells with Orthotropic Cores," M.S. thesis, Dec. 1973, Civil Engineering Dept., Marquette Univ., Milwaukee, Wis.

Convergence and Stability of Nonlinear Finite Element Equations

T. J. Chung*
The University of Alabama in Huntsville, Huntsville, Ala.

IN recent years, the field of computational mechanics has broadened significantly, particularly with the finite element analysis applied to solution of initial and/or boundary value problems in both solids and fluids. The question is, "Are we assured of convergence and stability in the solution?" In general, the finite element equations occur in the form

$$A_{ij}(u_j)u_j-f_i=0 \tag{1}$$

for the steady state, and

$$B_{ij}\dot{u}_j+C_{ij}(u_j)u_j-f_i=0 \tag{2}$$

for the unsteady state. Here the superposed dot denotes a time derivative.

We concern ourselves with the convergence criteria of Eq. (1) and both stability and convergence criteria of Eq. (2) in the solutions of these equations. The literature on the subject of convergence and stability is abundant; for example, Ortega and Rheinboldt[1] for nonlinear equations and Richtmyer and Morton[2] for time-dependent equations. Their discussions are concerned with approximate numerical solutions via finite difference equations. Subsequently, Oden,[3] Fujii,[4] and others studied the problems of convergence and stability associated with finite elements. The present study is intended for derivations of explicit convergence criteria for nonlinear finite element equations and stability criteria for linear and nonlinear time dependent finite element equations.

Consider the nonlinear finite element equations of the form

$$R_i(u_j)=A_{ij}(u_j)u_j-f_i=0 \tag{3}$$

Expanding Eq. (3) in Taylor series and retaining only the first-order terms yield

$$R_i(u_j)=R_i(u_j+\Delta u_j)=R_i(u^\circ_j) + [\partial R_i(u^\circ_j)/\partial u_j]\ (u_j-u^\circ_j)=0 \tag{4}$$

Solving for u_i in Eq. (4) gives

$$u_i=u^\circ_i-(J^\circ_{ij})^{-1}R_j(u^\circ_j) \tag{5}$$

where J°_{ij} is the Jacobian defined as

$$J^\circ_{ij}=\partial R_i(u^\circ_j)/\partial u_j \tag{6}$$

Received October 7, 1974; revision received January 17, 1975.
Index categories: Structural Dynamic Analysis, Subsonic and Transonic Flow.

*Professor of Mechanical Engineering. Member AIAA.

Assuming the initial values u_i, we obtain the successive solutions of u_i at $r+1$th step in the form

$$u_i^{r+1} = u_i^r - (J_{ij}^r)^{-1} R_j(u_j^r) \tag{7}$$

This is the well-known Newton-Raphson method of solving the nonlinear equations.

In the Newton-Raphson iterative process, we are concerned with the existence and uniqueness of the solution. To this end, the Jacobian matrix J_{ij}°, J_{ij}^r must be nonsingular. We are also concerned with convergence and the rate of convergence. If we differentiate Eq. (7) with respect to u_i^r, we observe that, for convergence, we must have

$$\partial u_i^{r+1}/\partial u_j^r = \delta_{ij} - \{ (J_{ik}^r)^{-1} J_{kj}^r - (J_{ik}^r)^{-1} (J_{k\ell}^r)^{-1} (\partial J_{\ell m}^r / \partial u_m^r) R_j(u_j^r) \} \leq \delta_{ij} \tag{8}$$

where δ_{ij} is the Kronecker delta. Rewriting Eq. (8), we obtain

$$(\partial u_i^{r+1}/\partial u_j^r) = (J_{ik}^r)^{-1} (J_{k\ell}^r)^{-1} (\partial J_{\ell m}^r/\partial u_m^r) R_j(u_j^r) \leq \delta_{ij} \tag{9}$$

Thus, the criterion for convergence is

$$(\partial J_{\ell m}^r / \partial u_m^r) R_j(u_j^r) \leq J_{ik}^r J_{k\ell}^r \delta_{ij} \tag{10}$$

The rate of convergence in this case may be evaluated as follows. Denote the error at the rth and $r+1$th steps as

$$e_i^r = u_i - u_i^r \tag{11}$$

$$e_i^{r+1} = u_i - u_i^{r+1} \tag{12}$$

Subtracting both sides of Eq. (7) from u_i, we get

$$u_i - u_i^{r+1} = u_i - u_i^r + \{ J_{ij}^r (u_i - e_i^r) \}^{-1} R_j(u_j^r) \tag{13}$$

In view of Eqs. (11-13) we have

$$e_i^{r+1} = e_i^r + \{ J_{ij}^r (u_i - e_i^r) \}^{-1} R_j(u_i - e_i^r) \tag{14}$$

Expanding the last terms on the right-hand side of Eq. (14) in Taylor series and neglecting higher order terms, we obtain

$$e_i^{r+1} = \tfrac{1}{2} e_i^r e_j^r \{ J_{jk}^r (u_i) \}^{-1} (\partial J_{k\ell} / \partial u_\ell) \tag{15}$$

or

$$e_i^{r+1} = 0\{ (e_i^r)^2 \} \tag{16}$$

This implies that the error decreases with the square of the error at the previous step. In other words, the rate of convergence is said to be second order. The finite element discretization errors are not included in the present study.

The finite element equations of unsteady state is given by

$$B_{ij} \dot{u}_j + C_{ij} u_j = 0 \tag{17}$$

where B_{ij} and C_{ij} are the constant coefficient matrices. A forward difference of the time derivative of u_j is

$$\dot{u}_j = (u_j^{n+1} - u_j^n)/\Delta t \tag{18}$$

where Δt represents $n+1$th time step, $\Delta t = t^{n+1} - t^n$. The difference operator of this form is called the temporal operator. Inserting Eq. (18) into Eq. (17) gives

$$u_j^{n+1} = u_j^n - B_{j\ell}^{-1} C_{\ell k} u_k^n \Delta t \tag{19}$$

If the errors at the $n+1$th step and nth step are given by e_j^{n+1} and e_j^n, respectively, we may write Eq. (19) in terms of these errors,

$$u_j^{n+1} + e_j^{n+1} = u_j^n + e_j^n - B_{j\ell}^{-1} C_{\ell k} (u_k^n + e_k^n) \Delta t \tag{20}$$

Comparing Eqs. (19) and (20) results in

$$e_j^{n+1} = (\delta_{jk} - B_{j\ell}^{-1} C_{\ell k} \Delta t) e_k^n \tag{21}$$

For the temporal operator of the form Eq. (18) to be successful the error for the $n+1$ step must not be permitted to grow, or

$$|e_j^{n+1}| \leq |e_k^n| \tag{22}$$

This requirement may be satisfied by

$$|g_{jk}| = |\delta_{jk} - B_{j\ell}^{-1} C_{\ell k} \Delta t| \leq |\delta_{jk}| \tag{23}$$

Here g_{ik} is called the amplification matrix and

$$|g_{ik}| = (g_{ik} g_{ik})^{1/2}$$

Rearranging Eq. (23) yields

$$-B_{j\ell}^{-1} C_{\ell k} \Delta t \geq -2\delta_{jk} \tag{24}$$

or

$$B_{j\ell}^{-1} C_{\ell k} \Delta t \leq 2\, \delta_{jk}$$

from which we obtain

$$\Delta t \leq |2 B_{ij} C_{k\ell}^{-1} \delta_{jk}| \tag{25}$$

It is concluded that the temporal operator Eq. (18) is conditionally stable. The stability is guaranteed if Eq. (25) is met.

If we replace u_j^n by $(u_j^{n+1} + u_j^n)/2$ then instead of Eq. (18) we have

$$B_{ij} \frac{(u_j^{n+1} - u_j^n)}{\Delta t} + C_{ij} \frac{(u_j^{n+1} + u_j^n)}{2} = 0$$

or

$$u_j^{n+1} - u_j^n = -\tfrac{1}{2} B_{j\ell}^{-1} C_{\ell k} (u_k^{n+1} + u_k^n)\, \Delta t$$

and

$$u_k^{n+1} = E_{j\ell}^{-1} F_{j\ell} u_k^n \tag{26}$$

where

$$E_{j\ell} = \delta_{j\ell} + \tfrac{1}{2} B_{jm}^{-1} C_{m\ell} \Delta t$$

$$F_{j\ell} = \delta_{j\ell} - \tfrac{1}{2} B_{jm}^{-1} C_{m\ell} \Delta t$$

The errors at $n+1$ and n are related by

$$e_k^{n+1} = g e_k^n \tag{27}$$

where g is the amplification factor,

$$g = E_{j\ell}^{-1} F_{j\ell} \leq 1 \tag{28}$$

It is seen that, for all values of Δt, we have $g \leq 1$ as long as $B_{jm}^{-1} C_{m\ell}$ is positive definite. In this case, the temporal operator is said to be unconditionally stable. Recall that u_j^{n+1} is involved in representation of the time independent term. Determination of u_j^{n+1} was explicit in Eq. (19), whereas an inverse of $E_{j\ell}$ is required for the case of the latter before u_j^{n+1} can be calculated, thus an implicit solution. The temporal operators of the former type are referred to as explicit scheme, and those of the latter type are known as implicit scheme.

Additional examples may be cited this time by Taylor series expansion of a variable. Note that Eq. (17) may be written as

$$\dot{u}_i = -B_{i\ell}^{-1} C_{\ell j} u_j \tag{29}$$

Expanding Eq. (29) in Taylor series with a second-order accuracy gives

$$u_j^{n+1} = u_j^n - \Delta t\, \dot{u}_j^n + \tfrac{1}{2}(\Delta t^2\, \ddot{u}_j^n)$$

or

$$u_j^{n+1} = g_{jk}\, u_k^n \tag{30}$$

with

$$g_{jk} = \delta_{jk} - \Delta t B_{j\ell}^{-1} C_{\ell k} + \tfrac{1}{2}\Delta t^2 B_{j\ell}^{-1} C_{\ell m} B_{mn}^{-1} C_{nk}$$

The stability requirement $g_{jk} \leq \delta_{jk}$ gives

$$\Delta t B_{j\ell}^{-1} C_{\ell k} - \tfrac{1}{2}\Delta t^2 B_{j\ell}^{-1} C_{\ell m} B_{mn}^{-1} C_{nk} - 2\delta_{jk} \leq 0$$

Solving for Δt yields

$$\Delta t \leq |1 \pm i\sqrt{3}|\, B_{\ell k} C_{k\ell}^{-1}$$

The limiting value is, thus

$$\Delta t \leq 2\, B_{\ell k} C_{k\ell}^{-1} \tag{31}$$

The temporal operator of the type Eq. (30) and the resulting Eq. (31) are considered as the finite element analog of the well-known Lax-Wendroff scheme.[5,6] This is also an explicit scheme.

As a last example, consider a Taylor series expansion in the form

$$u_j^{n+s} = u_j^n + \Delta u_j^n s + \frac{1}{2!}\,\Delta^3 u_j^n\, s(s+1) + \frac{1}{3!}\,\Delta^3 u_j^n s(s+1)(s+2)$$
$$+ \ldots + \frac{1}{n!}\,\Delta^m u_j^n s(s+1)(s+2)(s+3)\ldots(s+m-1) \tag{32}$$

in which s may be given by $s = t/\Delta t$. The time derivatives of u at the time step n are

$$\left[\frac{\partial u_j}{\partial t}\right]^{(n)} = \dot{u}_j^n = \frac{\partial u_j}{\partial s}\,\frac{\partial s}{\partial t} = \left[\frac{\partial u_j{}^{n+s}}{\partial s}\,\frac{\partial s}{\partial t}\right]_{s=0}$$
$$= \frac{1}{\Delta t}\sum_{i=1}^{n}\frac{1}{i}\,\Delta^i u_j^n \tag{33}$$

and

$$\left[\frac{\partial^2 u_j}{\partial t^2}\right]^{(n)} = \ddot{u}_j^n = \left[\frac{\partial \dot{u}_j^{n+s}}{\partial t}\right]_{s=0} = \frac{1}{\Delta t^2}\{\Delta^2 u_j^n$$
$$+ \Delta^3 u_j^n + \ldots + (2/m)\Delta^m u_j^n[1 + \tfrac{1}{2} + \ldots + 1/(m-1)]\} \tag{34}$$

where

$$\Delta u_j^n = u_j^n - u_j^{n-1}$$

$$\Delta^2 u_j^n = \Delta u_j^n - \Delta u_j^{n-1} = u_j^n - 2u_j^{n-1} + u_j^{n-2}$$

$$\Delta^3 u_j^n = \Delta^2 u_j^n - \Delta^2 u_j^{n-1} = u_j^n - 3u_j^{n-1} + 3u_j^{n-2} - u_j^{n-3}$$

Substituting this into Eq. (33) and (34) results in, respectively,

$$\dot{u}_j^n = (11\, u_j^n - 18\, u_j^{n-1} + 9\, u_j^{n-2} - 2\, u_j^{n-3})/6\Delta t \tag{35}$$

$$\ddot{u}_j^n = (2\, u_j^n - 5\, u_j^{n-1} + 4\, u_j^{n-2} - u_j^{n-3})/\Delta t^2 \tag{36}$$

In view of Eqs. (35) and (17) and moving up one step, we obtain

$$[B_{ij}(11\, u_j^{n+1} - 18 u_j^n + 9\, u_j^{n-1}$$
$$-2\, u_j^{n-2})/6\Delta t] + C_{ij} u_j{}^n = 0 \tag{37a}$$

or

$$[B_{ij}(11\, u_j^{n+1} - 18\, u_j^n + 9\, u_j^{n-1}$$
$$-2\, u_j^{n-2})/6\Delta t] + C_{ij} u_j^{n+1} = 0 \tag{37b}$$

$$[B_{ij}(11\, u_j^{n+1} - 18\, u_j^n + 9\, u_j^{n-1} - 2\, u_j^{n-2})/6\Delta t]$$
$$+ \tfrac{1}{2} C_{ij}(u_j^{n+1} + u_j^n) = 0 \tag{37c}$$

Note that the expression Eq. (37a) results in explicit scheme whereas the expressions Eqs.(37b) and (37c) are of the implicit scheme. If we assume that the errors at various time steps are related by

$$e_j^{n+1} \leq e_j^n \leq e_j^{n-1} \leq g_{jk}\, e_k^{n-2}$$

with $g_{jk} \leq \delta_{jk}$, then the amplifications are related as follows: for Eq. (37a)

$$g_{jk} = (18\alpha\delta_{jk} - 6\alpha B_{j\ell}^{-1} C_{\ell k}\Delta t$$
$$-9\beta\delta_{jk} + 2\delta_{jk})/11 \leq \delta_{jk} \tag{38a}$$

for Eq. (37b)

$$g = E_{j\ell}^{-1} F_{j\ell} \leq 1$$

$$E_{j\ell} = \delta_{j\ell} + (6/11) B_{jm}^{-1} C_{m\ell}\Delta t$$

$$F_{j\ell} = (18\alpha/11)\delta_{j\ell} - (9\beta/11)\delta_{j\ell} + (2/11)\delta_{j\ell} \tag{38b}$$

for Eq. (37c)

$$g = E_{j\ell}^{-1} F_{j\ell} \leq 1$$

$$E_{j\ell} = \delta_{j\ell} + (3/11) B_{jm}^{-1} C_{m\ell}\Delta t$$

$$F_{j\ell} = (18\alpha/11)(\delta_{j\ell} - (3/11) B_{jm}^{-1} C_{m\ell}\Delta t)$$
$$-(9\beta/11)\delta_{j\ell} + (2/11)\delta_{j\ell} \tag{38c}$$

where $\alpha \leq \beta \leq 1$. If we assume $\alpha = \beta = 1$, the expression Eq. (38a) becomes

$$g_{jk} = \delta_{jk} - (6/11) B_{j\ell}^{-1} C_{\ell k}\Delta t \tag{39}$$

and limiting Δt is

$$\Delta t \leq (12/11) B_{\ell k} C_{k\ell}^{-1} \tag{40}$$

Similarly the expressions (38 b,c) become, respectively,

$$g = [\delta_{j\ell} + (6/11) B_{jm} C_{m\ell}^{-1}\Delta t^{-1}\delta_{j\ell} \leq 1 \tag{41}$$

and

$$g = [\delta_{j\ell} + (3/11) B_{jm}\, C_{m\ell}^{-1}\Delta t]^{-1}[\delta_{j\ell} - (54/11) B_{jm} C_{m\ell}^{-1}\Delta t] \tag{42}$$

Once again, the last two cases are of implicit schemes and it is shown that, for all values of Δt, the temporal operators are unconditionally stable.

Let us now consider a nonlinear time-dependent finite element equation of the form

$$B_{ij}\dot{u}_j + C_{ij}(u_j)u_j = 0 \tag{43}$$

where the coefficient matrix C_{ij} is a function of the variable u_j. Take an implicit scheme Eq. (37b) and write in the form

$$R_i^{n+1}(u_j) = D_{ij}(u_j) u_j^{n+1} - f_i(u_j) = 0 \tag{44}$$

with

$$D_{ij} = \delta_{ij} + (6/11)\Delta t B_{i\ell}^{-1} C_{\ell j}) \tag{45}$$

$$f_i(u_j) = (18\alpha\delta_{ik} - 9\beta\delta_{ik} + 2\ \delta_{ik})u_k^{n-2}/11 \tag{46}$$

In view of Eq. (7) for each time increment, the Newton-Raphson iterative steps are governed by

$$u_j^{n+1,r+1} = u_j^{n+1,r} + (J_{ij}^{n+1,r})^{-1} R_j(u^{n+1,r}) \tag{47}$$

where

$$J_{ij}^{n+1} = \partial [R_i^{n+1}(u_j^r)] \leftrightarrow / \partial u_j \tag{48}$$

As given by Eq. (10), the convergence criterion is

$$(\partial J_{jm}^{n+1,r}/\partial u_m^r R_j^{n+1}(u_j^r) \leq J_{ik}^{n+1,r}\ J_{jk}{}^{n+1,r}\ \delta_{ij} \tag{49}$$

The rate of convergence is evaluated by noting the error at the $r+1$ th step as

$$r_i^{r+1} = \tfrac{1}{2} e_i^r\, e_j^r [J_{jk}^{n+1,r}(u_i)]^{-1} (\partial J_{k\ell}^r/\partial u_\ell) \tag{50}$$

It is seen that the Jacobian $J_{ij}{}^{n+1}$, which is the determining factor for convergence of the equations of nonlinear character, is affected by Δt since $R^{n+1}(u_j^r)$ is a function of Δt present in Eq. (44) or (45). It is clear, that the nonlinear time dependent finite element equations cannot be assured, unless Eqs. (38b) and (49) are simultaneously satisfied.

References

[1] Ortega, J. M. and Rheinboldt, W. C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[2] Richtmyer, R. D., and Morton, K. W., *Difference Methods of Initial Value Problems*, 2nd ed., Interscience Publishers, New York, 1967.

[3] Oden, J. T., *Finite Elements of Nonlinear Continua*, McGraw-Hill, New York, 1972.

[4] Fujii, H., "Finite Element Schemes: Stability and Convergence," *Advances in Computational Methods in Structural Mechanics and Design*, edited by J. T. Oden, R. W. Clough, and Y. Yamamoto, UAH Press, Huntsville, Ala., 1972, pp. 201-218.

[5] Lax, P. D. and Wendroff, B., "Systems of Conservation Laws," *Communications on Pure and Applied Mathematics*, Vol. 13,No.2, May 1960, p. 217-237.

[6] Lax, P. D. and Wendroff, B., "On Stability of Difference Schemes," *Communications on Pure and Applied Mathematics*, Vol. 15,No.4, Nov. 1962, pp. 363-371.

Prediction of Recovery Factor and Reynolds' Analogy for Compressible Turbulent Flow

Philip M. Gerhart*
University of Akron, Akron, Ohio

Introduction

RECENTLY the so-called surface renewal and penetration model of turbulent transport in the vicinity of a wall has been applied to a wide variety of flow problems.[1-6] Of particular interest to workers in the field of high-speed flows are the prediction of recovery factor $[R=(T_{aw}-T_\infty)/(T_{0\infty}-T_\infty)]$, and the effects of viscous dissipation on the Reynolds' analogy factor $(RAF=2St/C_f)$ in turbulent flows.

Received September 27, 1974; revision received November 20, 1974. This work was done while the author was a summer employee at NASA Lewis Research Center.

Index category: Boundary-Layer and Convective Heat Transfer—Turbulent.

*Assistant Professor of Mechanical Engineering.

In two recent papers, Thomas and Chung[5,6] have successfully applied the surface renewal model to the prediction of recovery factor and Reynolds' analogy factor including the effects of viscous dissipation; however, these analyses considered only the case of constant fluid properties, and hence their application to high-speed flows is not immediately obvious. It is the purpose of this Note to formulate the surface renewal and penetration model for compressible flows including viscous dissipation and to establish the approximate validity of the previous analyses for compressible gas flows.[5,6]

Analysis

The surface renewal and penetration model as first set forth by Danckwerts[7] is based on the assumption that macroscopic chunks of fluid ("eddies") intermittently move from the turbulent core into the close vicinity of the transport surface. During the time of residence in the wall region, unsteady one-dimensional molecular transport of momentum and energy are assumed to dominate. Several experimental investigations of incompressible turbulent flows are in basic agreement with this model,[8-10] except that the fluid elements do not move into direct contact with the wall; however, the assumption that they do reach the wall has been found to yield agreeable results for Prandtl numbers less than 10.[11]

Consider an eddy which moves from the turbulent core into contact with the wall. Neglecting axial gradients as small compared with stationary unsteady terms and transverse gradients, the continuity, momentum, and energy equations are

$$\frac{\partial \rho}{\partial \theta} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial \theta} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho\left(\frac{\partial h}{\partial \theta} + v\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \dagger \tag{3}$$

wherein θ is the instantaneous contact time, and a perfect gas $(h=C_pT)$ has been assumed. In Eqs. (2) and (3) the axial pressure gradient has been neglected. While nominally restricting the resulting analysis to flat plate flows, the pressure gradient term has, for incompressible flows, been shown to be negligible for tube flows above Reynolds numbers of 10^4(Ref. 2) and for boundary layers in mild pressure gradients.[3]

For flows not too near separation, the residence time of a typical eddy in the wall region is such that the eddy may be considered semi-infinite in the transverse direction.[3,4] It is also assumed that transfer of momentum and energy to the eddy during its flight from the turbulent core to the wall is negligible, thus the initial and boundary conditions on Eqs. (1-3) become

$$u=0, \quad h=h_w \quad \text{at } y=0 \tag{4a}$$

$$u=U_\infty, \quad h=h_\infty \quad \text{at } y\to\infty \tag{4b}$$

$$u=U_\infty, \quad h=h_\infty \quad \text{at } \theta=0 \tag{4c}$$

Equations (1-4) are now transformed via the introduction of

$$dY = (\rho/\rho_\infty)dy \tag{5}$$

resulting in

$$\frac{\partial u}{\partial \theta} = \nu_\infty \frac{\partial}{\partial Y}\left[C\frac{\partial u}{\partial Y}\right] \tag{6}$$

$$\frac{\partial T}{\partial \theta} = \frac{\nu_\infty}{Pr}\frac{\partial}{\partial Y}\left[C\frac{\partial T}{\partial Y}\right] + \frac{C\nu_\infty}{C_p}\left[\frac{\partial u}{\partial Y}\right]^2 \tag{7}$$

†Although the transverse convection terms may be neglected, the transformation to be introduced causes them to drop out identically.